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TYPE (2,1) REGULAR SUBMODELS OF THE EQUATIONS OF GAS DYNAMICS

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Partially invariant solutions (submodels) of type (2,1) are distinguished by the values of rank $\sigma = 2$ and defect $\delta = 1$. The regularity property of such solutions means that the leading subgroup (subalgebra), for which a partially invariant solution of type (2,1) is determined, has, in the basis space, exactly two functionally independent invariants, which are functions of independent variables [1]. For the equations of gas dynamics with the equation of state of the general form that admit the Lie algebra of the operators L_{11}, regular solutions of type (2,1) are possible only for some three-dimensional subalgebras $L_{3,i}$. The comprehensive list of such subalgebras is easily derived from the optimal system ΘL_{11} in [2]. This list includes 12 nonsimilar representatives, each of which generates a regular submodel of type (2,1). This work is concerned with describing all these submodels. Such classes of solutions have been previously considered only selectively [1, 3].

1. Preliminary Information. The equations of gas dynamics

$$\rho D\mathbf{u} + \nabla p = 0, \qquad D\rho + \rho \operatorname{div} \mathbf{u} = 0, \qquad DS = 0, \qquad p = F(\rho, S) \tag{1.1}$$

are considered in standard designations of time t, coordinates $\mathbf{x} = (x, y, z)$, and the required quantities: velocity vector $\mathbf{u} = (u, v, w)$, density ρ, pressure p, and entropy S. The symbol D denotes the substantial derivative $D = \partial_t + u\partial_x + v\partial_y + w\partial_z$. The function F gives the equation of state for a gas [the last in (1.1)] and is considered an arbitrarily fixed, rather smooth function, which satisfies the inequalities $F_\rho > 0$ and $F_S > 0$. The sound speed c, for which $c^2 = F_\rho$, and specific enthalpy $e = \int \rho^{-1} F_\rho \, d\rho$ are also used in description of the submodels.

The set of all submodels of type (2,1) for system (1.1) is a combination of the subsets of submodels of two characteristic classes: evolutionary χ^e and stationary χ^s (a similar subdivision was made in [4]). The invariant subsystem of equations in submodels of the class χ^e is qualitatively similarity to the system of equations of the invariant submodel of the one-dimensional, unsteady motion of a gas. A characteristic property of such systems is that they are of a hyperbolic type. The search for solutions in this case is reduced to one quasilinear second-order equation of hyperbolic type for the Lagrangian coordinate. The invariant subsystem of equations in submodels of the class χ^s has a qualitative similarity to the system of equations for two-dimensional steady gas flows. A characteristic property of such systems is their mixed elliptic-hyperbolic type. For these submodels, one can introduce the stream function and obtain Bernoulli and vorticity integrals.

In each submodel of type (2,1), only one sought-for "superfluous" function participates, which is not invariant and satisfies the overdetermined system of differential equations. Therefore the question of the existence of solutions of type (2,1) is nontrivial and is answered as the above-mentioned overdetermined system is reduced to passive form. This yields a definite invariant subsystem, whose solutions admit representation of some finite integrals.

The formation of the invariant subsystem is connected with the introduction of the desired auxiliary invariant function h, which is considered different from zero. In the case $h = 0$, the initial system is split into two subsystems, one of which determines the superfluous function and the other does not depend on this function and is invariant with respect to a certain two-dimensional subgroup. Thus, the entire solution

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is not reduced to the invariant solution, except for the submodel $\Pi(3, 15^{00})$. At the same time, the case of $h \neq 0$, where the superfluous function influences significantly the structure of the invariant subsystem, is most interesting. It is assumed, therefore, that the relationship $h \neq 0$ holds everywhere.

The submodels obtained with the help of the leading subalgebra $L_{3,i}$ are denoted by the symbol $\Pi(3, i)$, where i is the ordinal number of the three-dimensional subalgebra from [2, Table 6].

The submodels of the characteristic class χ^e are initially considered. There are seven such submodels. Their representation goes in increasing order of complexity.

2. $\Pi(3, 46)$. The basis of the subalgebra is $X_1 = \partial_x$, $X_2 = \partial_y$, and $X_4 = t\partial_x + \partial_u$. The invariants are t, z, v, w, ρ, p, and S. The "superfluous" function is u.

The representation of the solution is

$$(v, w, \rho, p, S) \mid (t, z), \qquad u = u(t, x, y, z).$$

The equations of the submodel are

$$u_t + uu_x + vu_y + wu_z = 0; \tag{2.1$_1$}$$

$$v_t + wv_z = 0; \tag{2.1$_2$}$$

$$w_t + ww_z + \rho^{-1}p_z = 0; \tag{2.1$_3$}$$

$$\rho_t + w\rho_z + \rho(u_x + w_z) = 0; \tag{2.1$_4$}$$

$$S_t + wS_z = 0. \tag{2.1$_5$}$$

From (2.1$_4$), it follows that $u_x = h$ is a function of only the variables t and z. Together with (2.1$_1$) this yields an overdetermined system for u:

$$u_x = h(t, z), \qquad u_t + vu_y + wu_z + hu = 0. \tag{2.2}$$

The compatibility condition for system (2.2) is

$$h_t + wh_z + h^2 = 0. \tag{2.3}$$

Equations (2.3), (2.1$_2$), (2.1$_3$), (2.1$_4$) (where $u_x = h$), and (2.1$_5$) form an invariant subsystem for the functions u, w, ρ, S, and h.

The Lagrangian coordinate $\xi = \xi(t, z)$ is introduced by the relations

$$\rho = h\xi_z, \qquad \rho w = -h\xi_t. \tag{2.4}$$

By virtue of these relations and (2.3), Eq. (2.1$_4$) is satisfied identically. In addition, (2.1$_2$) and (2.1$_5$) have the integrals $v = V(\xi)$, $S = S(\xi)$ and (2.3) is integrated in the form

$$h = 1/(t + T(\xi)) \tag{2.5}$$

with arbitrary functions $V(\xi)$, $S(\xi)$, and $T(\xi)$.

The overdetermined system (2.2) has the general solution

$$u = h(t, \xi)[x + X(\xi, \lambda)], \qquad \lambda = y - tV(\xi)$$

with an arbitrary function $X(\xi, \lambda)$.

The remaining equation (2.1$_3$) is reduced, with the aid of (2.4) and (2.5), to the equation for the Lagrangian coordinate ξ:

$$(\xi_t^2 - c^2\xi_t^2)\xi_{zz} - 2\xi_t\xi_z\xi_{zt} + \xi_z^2\xi_{tt} = \xi_z^2\left[h^{-1}F_S S'(\xi) - hc^2T'(\xi)\right]. \tag{2.6}$$

In Eq. (2.6) all the coefficients on the right-hand side are known functions of t, ξ, ξ_t, and ξ_z.

3. $\Pi(3, 38_1^{000})$ (from the series of subalgebras $L_{3,38}$ a sublagebra with $\alpha = \sigma = \tau = 0$, and $\beta = 1$ is selected). The basis of the subalgebra is $X_3 = \partial_z$, $X_1 + X_5 = \partial_x + t\partial_y + \partial_v$, and $X_6 = t\partial_z + \partial_w$. The invariants are t, $\lambda = x - y/t$, u, $V = v - y/t$, ρ, p, and S. The "superfluous" function is w.

The representation of the solution is

$$(\rho, p, S) \mid (t, \lambda), \quad u = U(t,\lambda) + t^{-1}V(t,\lambda), \quad v = y/t + V(t,\lambda), \quad w = W(t,\lambda,y,z).$$

The equations of the submodel are

$$U_t + UU_\lambda + (1+t^{-2})\rho^{-1}p_\lambda = 2t^{-2}V; \tag{3.1$_1$}$$

$$V_t + UV_\lambda + t^{-1}V - t^{-1}\rho^{-1}p_\lambda = 0; \tag{3.1$_2$}$$

$$W_t + UW_\lambda + (y/t + V)\,W_y + WW_z = 0; \tag{3.1$_3$}$$

$$\rho_t + U\rho_\lambda + \rho\left(U_\lambda + t^{-1} + W_z\right) = 0; \tag{3.1$_4$}$$

$$S_t + US_\lambda = 0. \tag{3.1$_5$}$$

From (3.1$_4$), it follows that $W_z = h$ is a function of only the variables t and λ. Together with (3.1$_3$), this yields an overdetermined system for W:

$$W_z = h(t,\lambda), \qquad W_t + UW_\lambda + (y/t + V)\,W_y + hW = 0. \tag{3.2}$$

The compatibility condition of system (3.2) is

$$h_t + Uh_\lambda + h^2 = 0. \tag{3.3}$$

Equations (3.3), (3.1$_1$), (3.1$_2$), (3.1$_4$) (where $W_z = h$), and (3.1$_5$) form an invariant subsystem for the functions u, V, ρ, S, and h.

The Lagrangian coordinate $\xi = \xi(t,\lambda)$ is introduced by the relations

$$t\rho = h\xi_\lambda, \qquad t\rho U = -h\xi_t \tag{3.4}$$

By virtue of the latter and (3.3), Eq. (3.1$_4$) is satisfied identically. In addition, the combination of (3.1$_1$) and (3.1$_2$) and also the combination of (3.1$_5$) and (3.3) have the integrals

$$U + (t + t^{-1})V = H(\xi), \quad S = S(\xi), \qquad h = 1\big/(t + T(\xi)) \tag{3.5}$$

with arbitrary functions $H(\xi)$, $S(\xi)$, and $T(\xi)$.

The overdetermined system (3.2) has the general solution

$$W = h(t,\xi)[z + Z(\xi,j)], \qquad j = y/t - \int t^{-1}V(t,\xi)\,dt$$

with an arbitrary function $Z(\xi, j)$.

The remaining Eq. (3.1$_1$) is reduced, using (3.4) and (3.5), to the equation for the Lagrangian coordinate $\xi(t,\lambda)$, which is similar to (2.6).

4. $\Pi(3, 38_2^{000})$ (from the series of subalgebras $L_{3,38}$ a subalgebra with $\alpha = \beta = \sigma = 0$ and $\tau = 1$ is selected). The basis of the subalgebra is $X_3 = \partial_z$, $X_5 = t\partial_y + \partial_v$, and $X_2 + X_6 = \partial_y + t\partial_z + \partial_w$. The invariants are t, x, u, $W = w + tv - y$, ρ, p, and S. The "superfluous" function is v.

The representation of the solution is

$$(u, \rho, p, S) \mid (t, x), \qquad w = y - tv + W(t,x), \qquad v = v(t,x,y,z).$$

The equations of the submodel are

$$u_t + uu_x + \rho^{-1}p_x = 0; \tag{4.1$_1$}$$

$$v_t + uv_x + vv_y + (y - tv + W)v_z = 0; \tag{4.1$_2$}$$

$$W_t + uW_x = 0; \tag{4.1$_3$}$$

$$\rho_t + u\rho_x + \rho u_x + \rho(v_y - tv_z) = 0; \tag{4.1$_4$}$$

$$S_t + uS_x = 0. \tag{4.1$_5$}$$

From (4.1$_4$), it follows that $v_y - tv_z = h$ is a function of only the variables t and x. Together with Eq. (4.1$_2$) this yields an overdetermined system for v:

$$v_y - tv_z = h(t,x), \qquad v_t + uv_x + (y+W)v_z + hv = 0. \tag{4.2}$$

Reduction of this system to involution is performed in "two steps" and gives the following result. With the new auxiliary function $k = k(t,x)$ the passive form of system (4.2) is

$$v_z = k, \qquad v_y = tk + h, \qquad v_t + uv_x + hv = -(y+W)k, \tag{4.3}$$

where the functions h and k are related by the equations

$$h_t + uh_x + h^2 + 2k = 0, \qquad k_t + uk_x + hk = 0. \tag{4.4}$$

Equations (4.4), (4.1$_1$), (4.1$_3$), (4.1$_4$) (where $v_y - tv_z = h$), and (4.1$_5$) form an invariant subsystem for the functions u, W, ρ, S, h, and k.

The Lagrangian coordinate $\xi = \xi(t,x)$ is introduced by the relations

$$\rho = k\xi_x, \qquad \rho u = -k\xi_t. \tag{4.5}$$

By virtue of this and (4.4), Eq. (4.1$_4$) is satisfied identically. In addition, Eqs. (4.1$_3$) and (4.1$_5$) have the integrals $W = W(\xi)$ and $S = S(\xi)$ with arbitrary functions $W(\xi)$ and $S(\xi)$.

The general solution of the overdetermined system (4.3) is found in the form

$$v = (z+ty)k + yh + V(t,x),$$

where the function $V(t,x)$ is the general solution of the equation

$$V_t + uV_x + hV + kW = 0. \tag{4.6}$$

After transformation to the Lagrangian coordinates t and ξ the system (4.4), (4.6) is transformed into a system of ordinary differential equations:

$$h_t + h^2 + 2k = 0, \qquad k_t + hk = 0, \qquad V_t + hV + kW = 0,$$

which are integrated explicitly. The general solution of the subsystem (4.4), (4.6) is

$$h = \frac{2At+B}{At^2+Bt+C}, \qquad k = -\frac{A}{At^2+Bt+C}, \qquad V = W\frac{At+D}{At^2+Bt+C}, \tag{4.7}$$

where A, B, C, and D are arbitrary functions of ξ.

The remaining equation (4.1$_1$) is reduced with the aid of (4.5) and (4.7) to an equation for the Lagrangian coordinate $\xi(t,x)$, which is similar to (2.6).

5. $\Pi(3,15^{00})$ (from the series of subalgebras $L_{3,15}$ a subalgebra with $\alpha = \beta = 0$ is selected). The basis of the subalgebra is $X_3 + X_5 = \partial_z + t\partial_y + \partial_v$, $X_2 - X_6 = \partial_y - t\partial_z - \partial_w$, and $X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v$. The invariants are t, x, u, V, ρ, p, and S, where

$$V = \sqrt{\left(v - \frac{ty+z}{t^2+1}\right)^2 + \left(w - \frac{tz-y}{t^2+1}\right)^2}.$$

The function θ, which appears in the expressions

$$v = \frac{ty+z}{t^2+1} + V\cos\theta, \qquad w = \frac{tz-y}{t^2+1} + V\sin\theta,$$

is "superfluous."

The solution is represented as

$$(u, V, \rho, p, S) \mid (t,x), \qquad \theta = \theta(t,x,y,z).$$

The submodel has the equations

$$u_t + uu_x + \rho^{-1}p_x = 0; \tag{5.1$_1$}$$

$$V_t + uV_x + \frac{tV}{t^2+1} = 0; \tag{5.1$_2$}$$

$$\theta_t + u\theta_x + \left(V\cos\theta + \frac{ty+z}{t^2+1}\right)\theta_y + \left(V\sin\theta + \frac{tz-y}{t^2+1}\right)\theta_z = \frac{1}{t^2+1}; \tag{5.1$_3$}$$

$$\rho_t + u\rho_x + \rho\left(u_x + \frac{2t}{t^2+1} - V\sin\theta\,\theta_y + V\cos\theta\,\theta_z\right) = 0; \tag{5.1$_4$}$$

$$S_t + uS_x = 0. \tag{5.1$_5$}$$

From (5.1$_4$), it follows that $\sin\theta\,\theta_y - \cos\theta\,\theta_z = h$ is a function of only the variables t and x. Together with (5.1$_3$), this leads to an overdetermined system for θ:

$$\begin{gathered}\sin\theta\,\theta_y - \cos\theta\,\theta_z = h(t,x),\\ \theta_t + u\theta_x + \left(V\cos\theta + \frac{ty+z}{t^2+1}\right)\theta_y + \left(V\sin\theta + \frac{tz-y}{t^2+1}\right)\theta_z = \frac{1}{t^2+1}.\end{gathered} \tag{5.2}$$

Reduction of system (5.2) to involution shows that it is necessary that $h = 0$ and $\theta_y = \theta_z = 0$. This means that the submodel $\Pi(3, 15^{00})$ is reduced to an invariant submodel of type (2,0) with respect to the subalgebra $L_{2,21}$ with the basis $X_3 + X_5$, $X_2 - X_6$. A detailed analysis is given in the diploma work of S. V. Golovin.

6. $\Pi(3,13)$. The basis of the subalgebra is $X_2 = \partial_y$, $X_3 = \partial_z$, and $X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v$. The invariants are t, x, u, $q = \sqrt{v^2+w^2}$, ρ, p, and S. The "superfluous" function is $\theta = \arctan(w/v)$.

The solution is represented as

$$(u, q, \rho, p, S) \mid (t, x), \qquad v = q\cos\theta, \qquad w = q\sin\theta, \qquad \theta = \theta(t,x,y,z).$$

The submodel has the equations

$$u_t + uu_x + \rho^{-1}p_x = 0; \tag{6.1$_1$}$$

$$q_t + uq_x = 0; \tag{6.1$_2$}$$

$$\theta_t + u\theta_x + q\cos\theta\,\theta_y + q\sin\theta\,\theta_z = 0; \tag{6.1$_3$}$$

$$\rho_t + u\rho_x + \rho u_x + \rho q(-\sin\theta\,\theta_y + \cos\theta\,\theta_z) = 0; \tag{6.1$_4$}$$

$$S_t + uS_x = 0. \tag{6.1$_5$}$$

From (6.1$_4$), it follows that $-\sin\theta\,\theta_y + \cos\theta\,\theta_z = h$ is a function of only the variables t and x. Together with (6.1$_3$), this yields an overdetermined system for θ:

$$-\sin\theta\,\theta_y + \cos\theta\,\theta_z = h(t,x), \qquad \theta_t + u\theta_x + q\cos\theta\,\theta_y + q\sin\theta\,\theta_z = 0. \tag{6.2}$$

The compatibility condition for system (6.2) is

$$h_t + uh_x + h^2 q = 0. \tag{6.3}$$

Equations (6.3), (6.1$_1$), (6.1$_2$), and (6.1$_4$) (where h), and 6.1$_5$ form an invariant subsystem for the functions u, q, ρ, S, and h.

The Lagrangian coordinate $\xi = \xi(t,x)$ is introduced by the relations

$$\rho = h\xi_x, \quad \rho u = -h\xi_t. \tag{6.4}$$

By virtue of these relations and (6.3), Eq. (6.1$_4$) is satisfied identically. In addition, Eqs. (6.1$_2$), (6.1$_5$), and also (6.3) have the integrals

$$q = Q(\xi), \qquad S = S(\xi), \qquad h = 1/(tQ(\xi) + H(\xi)) \tag{6.5}$$

with arbitrary functions $Q(\xi)$, $S(\xi)$, and $H(\xi)$.

The overdetermined system (6.2) has a general solution, which is explicitly determined by the equation

$$f(\xi, \lambda, \mu) = 0, \qquad \lambda = y - h^{-1}\cos\theta, \qquad \mu = z - h^{-1}\sin\theta$$

with an arbitrary function f, which ensures the possibility of determining θ.

The remaining Eq. (6.1_1) is reduced with the aid of (6.4) and (6.5) to an equation for the Lagrangian coordinate $\xi(t,x)$, which is similar to (2.6).

7. $\Pi(3,11)$. The basis of the subalgebra is $X_1 = \partial_x$, $X_4 = t\partial_x + \partial_u$, and $X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v$. Cylindrical coordinates are introduced by the relations

$$y = R\cos\theta, \quad z = R\sin\theta, \quad v = V\cos\theta - W\sin\theta, \quad w = V\sin\theta + W\cos\theta. \tag{7.1}$$

The invariants are t, r, V, W, ρ, p, and S. The "superfluous" function is u.

The representation of the solution is

$$(V, W, \rho, p, S) \mid (t, r), \qquad u = u(t, x, r, \theta).$$

The equations of the submodel are

$$u_t + uu_x + Vu_r + r^{-1}Wu_\theta = 0; \tag{7.2$_1$}$$

$$V_t + VV_r + \rho^{-1}p_x = r^{-1}W^2; \tag{7.2$_2$}$$

$$W_t + VW_r = -r^{-1}VW; \tag{7.2$_3$}$$

$$\rho_t + V\rho_r + \rho\left(u_x + V_r + r^{-1}V\right) = 0; \tag{7.2$_4$}$$

$$S_t + VS_r = 0. \tag{7.2$_5$}$$

From (7.2_4), it follows that $u_x = h$ depends only on the variables t and r. Together with Eq. (7.2_1), this leads to an overdetermined system for u:

$$u_x = h(t, r), \qquad u_t + Vu_r + r^{-1}Wu_\theta + hu = 0. \tag{7.3}$$

The compatibility condition for system (7.3) is

$$h_t + Vh_r + h^2 = 0. \tag{7.4}$$

Equations (7.4), (7.2_2), (7.2_3), (7.2_4) (where $u_x = h$), and (7.2_5) form an invariant subsystem for the functions V, W, ρ, S, and h.

The Lagrangian coordinate $\xi = \xi(t,r)$ is introduced by the relations

$$r\rho = h\xi_r, \qquad r\rho V = -h\xi_t. \tag{7.5}$$

By virtue of these relations together with (7.4), Eq. (7.2_4) is satisfied. In addiiton, Eqs. (7.2_3), (7.2_5), and also (7.4) have the integrals

$$rW = Q(\xi), \qquad S = S(\xi), \qquad h = 1/(t + T(\xi)) \tag{7.6}$$

with arbitrary functions $Q(\xi)$, $S(\xi)$, and $T(\xi)$.

The overdetermined system (7.3) has the general solution

$$u = h(t,\xi)[x + X(\xi,\eta)], \qquad \eta = \theta - Q(\xi)\int \frac{dt}{r^2(t,\xi)}$$

with the arbitrary function $X(\xi,\eta)$.

The remaining equation (7.2_2) is reduced, using (7.5) and (7.6), to an equation for the Lagrangian coordinate $\xi(t,r)$, which is similar to (2.6).

8. $\Pi(3,8)$. The basis of the subalgebra is formed by the rotation operators X_7, X_8, and X_9. The problem is considered in spherical coordinates. Its solution is discussed in [5].

Thus, the list of type (2,1) regular submodels of the class χ^e is completed. Further we consider submodels of the class χ^s. There are five such submodels. We shall consider these submodels in increasing order of complexity.

9. $\Pi(3,29)$. The basis of the subalgebra is $X_1 = \partial_x$, $X_4 = t\partial_x + \partial_u$, and $X_{10} = \partial_t$. The invariants are y, z, v, w, ρ, p, and S. The "superfluous" function is u.

The representation of the solution is

$$(v, w, \rho, p, S) \mid (y, z), \qquad u = u(t, x, y, z).$$

The equations of the submodel are

$$u_t + uu_x + vu_y + wu_z = 0; \tag{9.1$_1$}$$

$$vv_y + wv_z + \rho^{-1}p_y = 0; \tag{9.1$_2$}$$

$$vw_y + ww_z + \rho^{-1}p_z = 0; \tag{9.1$_3$}$$

$$v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) = 0; \tag{9.1$_4$}$$

$$vS_y + wS_z = 0. \tag{9.1$_5$}$$

From (9.1$_4$), it follows that $u_x = h$ is a function of only the variables y and z. Together with Eq. (9.1$_1$), this yields an overdetermined system for u:

$$u_x = h(y, z), \qquad u_t + vu_y + wu_z + hu = 0. \tag{9.2}$$

The compatibility condition for system (9.2) is

$$vh_y + wh_z + h^2 = 0. \tag{9.3}$$

Equations (9.3), (9.1$_2$), (9.1$_3$), (9.1$_4$) (where $u_x = h$), and (9.1$_5$) form an invariant subsystem for the functions v, w, ρ, S, and h.

The stream function $\psi = \psi(y, z)$ is introduced by the relations

$$\rho v = h\psi_z, \qquad \rho w = -h\psi_y. \tag{9.4}$$

By virtue of these relations and (9.3), Eq. (9.1$_4$) is automatically satisfied. In addiiton, (9.1$_5$) has the integral $S = S(\psi)$ with an arbitrary function $S(\psi)$, and from (9.1$_2$) and (9.1$_3$) the Bernoulli integral follows:

$$v^2 + w^2 + 2e(\rho, S) = 2b(\psi) \tag{9.5}$$

with an arbitrary function $b(\psi)$ and enthalpy e. Also, eliminating pressure from Eqs. (9.1$_2$) and (9.1$_3$), we obtain the vorticity integral for the quantity $v_z - w_y$:

$$h(v_z - w_y) = F_\rho(\rho, S) + \rho\,[G(\psi) - e_S(\rho, S)] \tag{9.6}$$

where $G(\psi)$ is an arbitrary function.

The general solution of the overdetermined system (9.2) is given by the formula

$$u = h(y, z)(x + X(t, y, z)),$$

where the function $X = X(t, y, z)$ is found as the general solution of the equation

$$X_t + vX_y + wX_z = 0.$$

For the stream function $\psi(y, z)$ from (9.4)–(9.6), it is possible to obtain one quasilinear second-order equation, which also contains the unknown function h. Therefore, the above-mentioned equation should be solved together with (9.3).

10. $\Pi(3, 27^{00})$. The basis of the subalgebra is $X_3 = \partial_z$, $X_6 = t\partial_z + \partial_w$, and $X_4 + X_{10} = t\partial_x + \partial_u + \partial_t$. The invariants are $\lambda = x - (1/2)t^2$, y, $U = u - t$, v, ρ, p, and S. The "superfluous" function is w.

The representation of the solution is

$$u = t + U(\lambda, y), \qquad v = V(\lambda, y), \qquad (\rho, p, S) \mid (\lambda, y), \qquad w = w(t, \lambda, y, z).$$

The equations of the submodel are

$$UU_\lambda + VU_y + \rho^{-1}p_\lambda = -1,$$

$$UV_\lambda + VV_y + \rho^{-1}p_y = 0,$$

$$w_t + Uw_\lambda + Vw_y + ww_z = 0, \tag{10.1}$$
$$U\rho_\lambda + V\rho_y + \rho(U_\lambda + V_y + w_z) = 0,$$
$$US_\lambda + VS_y = 0.$$

System (10.1) coincides with system (9.1) (with accuracy to designations), except for the first equation of (10.1), which, in contrast to Eq. (9.1$_2$), has nonzero right-hand side (-1). Therefore all the relations obtained in Section 9 are valid, except for the Bernoulli integral, which in this case has the form

$$U^2 + V^2 + 2\lambda + 2e(\rho, S) = 2b(\psi).$$

11. $\Pi(3, 23)$. The basis of the subalgebra is $X_1 = \partial_x$, $X_4 = t\partial_x + \partial_u$, and $\alpha X_6 + X_{11} = \alpha(t\partial_z + \partial_w) + t\partial_t + x\partial_x + y\partial_y + z\partial_z$. The invariants are $\lambda = y/t$, $\mu = z/t - \alpha \ln t$, v, $w - z/t$, ρ, p, and S. The "superfluous" function is u.

The representation of the solution is

$$u = u(t, x, \lambda, \mu), \quad v = \lambda + V(\lambda, \mu), \quad w = \alpha + z/t + W(\lambda, \mu), \quad (\rho, p, S) \mid (\lambda, \mu).$$

The equations of the submodel are

$$t(u_t + uu_x) + Vu_\lambda + Wu_\mu = 0; \tag{11.1$_1$}$$
$$VV_\lambda + WV_\mu + V + \rho^{-1}p_\lambda = 0; \tag{11.1$_2$}$$
$$VW_\lambda + WW_\mu + W + \rho^{-1}p_\mu = -\alpha; \tag{11.1$_3$}$$
$$V\rho_\lambda + W\rho_\mu + \rho(tu_x + V_\lambda + W_\mu + 2) = 0; \tag{11.1$_4$}$$
$$VS_\lambda + WS_\mu = 0. \tag{11.1$_5$}$$

From Eq. (11.1$_4$), it follows that $tu_x = h$ is a function of only the variables λ and μ. Together with (11.1$_1$), this leads to an overdetermined system for u:

$$tu_x = h(\lambda, \mu), \qquad tu_t + Vu_\lambda + Wu_\mu + hu = 0. \tag{11.2}$$

The compatibility condition for system (11.2) is

$$Vh_\lambda + Wh_\mu = h - h^2. \tag{11.3}$$

Equations (11.3), (11.1$_2$), (11.1$_3$), (11.1$_4$) (where $tu_x = h$), and (11.1$_5$) form an invariant subsystem for the functions V, W, ρ, S, and h.

The stream function $\psi = \psi(\lambda, \mu)$ is introduced by the relations

$$\rho V = \frac{(h-1)^3}{h^2}\psi_\mu, \qquad \rho W = -\frac{(h-1)^3}{h^2}\,\psi_\lambda;$$

by virtue of these relations and in view of (11.3), Eq. (11.1$_4$) is satisfied identically. In addition, (11.1$_5$) has the entropy integral $S = S(\psi)$ with an arbitrary function $S(\psi)$.

The general solution of the overdetermined system (11.2) has the form

$$u = t^{-1}h(\lambda, \mu)[x + X(t, \lambda, \mu)],$$

where the function $X = X(t, \lambda, \mu)$ is sought for as a general solution of the equation

$$tX_t + VX_\lambda + WX_\mu = 0.$$

The Bernoulli and vorticity integrals are not obtained here. A particular case of this submodel is $\Pi(3, 24)$, obtained at $\alpha = 0$.

12. $\Pi(3, 17)$. The basis of the subalgebra is $X_1 = \partial_x$, $X_4 = t\partial_x + \partial_u$, and $X_7 + X_{10} = y\partial_z - z\partial_y + v\partial_w - w\partial_v + \partial_t$. The submodel is considered in cylindrical coordinates (7.1). The invariants are r, $\lambda = \theta - t$, V, W, ρ, p, S. The "superfluous" function is u.

The representation of the solution is

$$u = u(t, x, r, \lambda), \qquad (V, W, \rho, p, S) \mid (r, \lambda).$$

The equations of the submodel [after the replacement $W \to r(1+W)$] are

$$u_t + uu_x + Vu_r + Wu_\lambda = 0; \tag{12.1$_1$}$$

$$VV_r + WV_\lambda + \rho^{-1}p_r = r(1+W)^2; \tag{12.1$_2$}$$

$$r^2(VW_r + WW_\lambda) + \rho^{-1}p_\lambda = -2rV(1+W); \tag{12.1$_3$}$$

$$V\rho_r + W\rho_\lambda + \rho(u_x + V_r + r^{-1}V + W_\lambda) = 0; \tag{12.1$_4$}$$

$$VS_r + WS_\lambda = 0. \tag{12.1$_5$}$$

From (12.1$_4$), it follows that $u_x = h$ depends only on the variables r and λ. Together with (12.1$_1$), this leads to an overdetermined system for u:

$$u_x = h(r, \lambda), \qquad u_t + Vu_r + Wu_\lambda + hu = 0. \tag{12.2}$$

The compatibility condition for system (12.2) is

$$Vh_r + Wh_\lambda + h^2 = 0. \tag{12.3}$$

Equations (12.3), (12.1$_2$), (12.1$_3$), (12.1$_4$) (where $u_x = h$), and (12.1$_5$) form an invariant subsystem for the functions V, W, ρ, S, and h.

The stream function $\psi = \psi(r, \lambda)$ is introduced by the relations

$$r\rho V = h\psi_\lambda, \qquad r\rho W = -h\psi_r,$$

by virtue of which together with (12.3) Eq. (12.1$_4$) is satisfied identically. In addition, (12.1$_5$) has the entropy integral $S = S(\psi)$ with an arbitrary function $S(\psi)$.

The general solution of the overdetermined system (12.2) has the form

$$u = h(r, \lambda)[x + X(t, r, \lambda)],$$

where the function $X = X(t, r, \lambda)$ is a general solution of the equation

$$X_t + VX_r + WX_\lambda = 0.$$

From Eqs. (12.1$_2$) and (12.1$_3$), the Bernoulli integral

$$V^2 + r^2W^2 - r^2 + 2e(\rho, S) = 2b(\psi)$$

is obtained with an arbitrary function $b(\psi)$. The vorticity integral is found in the form

$$\frac{h}{r\rho}[V_\lambda - (r^2(1+W))_r] = -S'(\psi)\int \frac{F_S}{\rho^2}d\rho + G(\psi)$$

with an arbitrary function $G(\psi)$.

13. $\Pi(3,6)$. The basis of the subalgebra is $X_1 = \partial_x$, $X_4 = t\partial_x + \partial_u$, and $\beta X_7 + X_{11} = \beta(y\partial_z - z\partial_y + v\partial_w - w\partial_v) + t\partial_t + x\partial_x + y\partial_y + z\partial_z$. The submodel is considered in cylindrical coordinates (7.1). The invariants are $\lambda = r/t$, $\mu = \theta - \beta \ln t$, V, W, ρ, p, and S. The "superfluous" function is u.

The representation of the solution is

$$u = u(t, x, \lambda, \mu), \qquad (V, W, \rho, p, S) \mid (\lambda, \mu).$$

The equations of the submodel (after replacement $V \to \lambda + V$ and $W \to \lambda W + \beta\lambda$) are

$$t(u_t + uu_x) + Vu_\lambda + Wu_\mu = 0; \tag{13.1$_1$}$$

$$VV_\lambda + WV_\mu + V + \rho^{-1}p_\lambda = \lambda(W+\beta)^2; \tag{13.1$_2$}$$

$$\lambda^2(VW_\lambda + WW_\mu) + \lambda V(W+\beta) + \rho^{-1}p_\mu = -\lambda(\lambda + V)(W+\beta); \tag{13.1$_3$}$$

$$V\rho_\lambda + W\rho_\mu + \rho(tu_x + V_\lambda + \lambda^{-1}V + W_\mu + 2) = 0; \tag{13.1$_4$}$$

$$VS_\lambda + WS_\mu = 0. \tag{13.1$_5$}$$

From (13.1$_4$), it follows that $tu_x = h$ is a function of only the variables λ and μ. Together with (13.1$_1$), this yields an overdetermined system for u:

$$tu_x = h(\lambda, \mu), \qquad tu_t + Vu_\lambda + Wu_\mu + hu = 0. \tag{13.2}$$

The compatibility condition for system (13.2) is

$$Vh_\lambda + Wh_\mu = h - h^2. \tag{13.3}$$

Equations (13.3), (13.1$_2$), (13.1$_3$), (13.1$_4$) (where $tu_x = h$), and (13.1$_5$) form an invariant subsystem for the functions V, W, ρ, S, and h.

The stream function $\psi = \psi(\lambda, \mu)$ is introduced by the relations

$$\lambda\rho V = \frac{(h-1)^3}{h^2}\psi_\mu, \qquad \lambda\rho W = -\frac{(h-1)^3}{h^2}\psi_\lambda.$$

By virtue of these relationships and (13.3), Eq. (13.1$_4$) is satisfied identically. In addition, (13.1$_5$) has the entropy integral $S = S(\psi)$ with an arbitrary function $S(\psi)$.

The general solution of the redefined system (13.2) has the form

$$u = t^{-1}h(\lambda, \mu)[x + X(t, \lambda, \mu)],$$

where $X = X(t, \lambda, \mu)$ is a general solution of the equation

$$tX_t + VX_\lambda + WX_\mu = 0.$$

The Bernoulli and vorticity integrals cannot be obtained in this case.

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